it follows that $h_i/h_{ie} = (T/T_e)^a$, which yields

$$\frac{T}{T_e} = \left[\frac{\lambda + (1-\lambda)\phi - (u_e^2/2H_e)f'^2}{\sum_i \Upsilon_{ie} z_i \frac{h_{ie}}{H_e}}\right]^{1/a} \quad (38)$$

$$\frac{\rho_e}{\rho} = \frac{T}{T_e}\left(\frac{\sum_i \frac{\Upsilon_{ie} z_i}{W_i}}{\sum_i \frac{\Upsilon_{ie}}{W_i}}\right) \quad (39)$$

Equation (39), when evaluated at the reference condition, yields the density ratio required to determine κ from Eq. (23). Furthermore, since $(\rho_e/\rho_j) = f_j'^2$, the stagnation enthalpy ratio at the inner boundary $(\chi \rightarrow -\infty)$ cannot be chosen arbitrarily but must satisfy the relation

$$g_i = \frac{H_j}{H_e} = \left(f_j'^2 \frac{W_j}{W_e}\right)^a \sum_i \Upsilon_{ie} z_{ij} \frac{h_{ie}}{H_e} + \frac{u_e^2}{2H_e} f_j'^2 \quad (40)$$

For the frozen-flow case, the species mass fractions follow directly from Eq. (31) once $\phi(\chi)$ has been determined from the solution of Eqs. (29) and (30). The other flow property ratios of interest, namely h/he, T/Te, and ρ_e/ρ, are evaluated from Eqs. (37–39), respectively. The normal coordinate in the physical plane is related to χ by

$$y = \left(\frac{\beta k}{m\phi_*}\right)^{1/2} \frac{\bar{\rho}_e}{\rho_e} \left(\frac{u_e}{He^{1/2}}\right)^{\kappa-1} \frac{\bar{x}}{\sigma} \times \left[\kappa \int_0^\chi \phi d\chi' + (1-\kappa)\int_0^\chi f'^2 d\chi'\right] \quad (41)$$

Hence the transformation to the physical plane is complete once $\bar{x}$ is determined from Eq. (35).

For the equilibrium flow case, Eq. (31) can be used to determine the element mass fractions (see Ref. 3 for details) since, for this case, $z_i = z_k = \theta_k/\theta_{ke}$, where the element mass fractions θ_k are related to the species mass fractions by

$$\theta_k = \sum_{i=1}^{s} \alpha_{ik} \frac{W_k}{W_i} \Upsilon_i \quad (42)$$

Consequently, it is necessary to evaluate the species mass fractions and the temperature from the simultaneous solution of Eqs. (38) and (42), together with the relation for the equilibrium constants

$$K_{pl} = (pW)^{\sum_i \nu_{il}} \prod_{i=1}^{s} \left(\frac{\Upsilon_i}{W_i}\right)^{\nu_{il}} \quad (43)$$

In Eq. (43), the l subscript refers to the independent chemical reactions considered. The remainder of the inverse transformation follows a similar procedure as for the frozen-flow case once the species mass fractions and temperature have been determined.

It should be pointed out that, for temperatures of interest in supersonic combustion, it is possible to approximate the equilibrium behavior within the mixing region by adopting a flame-sheet combustion mode, as discussed in Ref. 4. This combustion model, which is based on the application of the low-speed diffusion flame concepts to high-speed flows, assumes that the reaction zone is of zero thickness and that the reacting species burn instantaneously upon reaching the sheet. These simplifications are justified provided the reaction rates are very large compared to the diffusion velocities of the species toward the sheet. The formulation and results of this approach can be found in Ref. 4.

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Deflections and Bending Moments of Rectangular Sandwich Panels with Clamped Edges under Combined Biaxial Compressions and Pressure

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Nomenclature

a, b, c	= length of panel; width of panel; core height
t_f	= face thickness
D_0	= flexural rigidity of panel
D_f	= flexural rigidity of one face sheet
G_c'	= shear stiffness of core
i, m, n	= odd integers unless noted
M_x, M_y	= sectional moments perpendicular to x axis and y axis, respectively
M_{xy}	= torsional moment per unit length of section perpendicular to x axis and y axes
N_x, N_y	= sectional loads perpendicular to x axis and y axes, respectively
p	= lateral pressure
W_{b1}	= bending deflection due to edge moment M_x
W_{b2}	= bending deflection due to edge moment M_y
$W_{b_{ss}}$	= bending deflection for simply supported edges
$W_{s_{ss}}$	= shear deflection for simply supported edges
$W_{T_{ss}} = W_{b_{ss}} + W_{s_{ss}}$	= total deflection for simply supported edges
x, y	= coordinates parallel to the length and width of the panel with origin at the center
X_n, Y_m	= function of x only, function of y only
∇^2	$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$
∇^4	$= \nabla^2 \nabla^2$
ϕ_m	$= m\pi/a$
β_n	$= \frac{n\pi}{b}$
μ	= Poisson's ratio

Received August 28, 1964; revision received December 7, 1964. The authors wish to thank North American Aviation, Inc., for permission to publish this result. Our expression of gratitude is extended to P. L. Brosseau, C. Brockway, and Florence Anderson for their help in Fortran programing, and to Jack Joanides and Howard Arvin who have given many helpful suggestions.

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Introduction

PAPERS on sandwich panels with clamped edges are few and are concerned mainly with the critical compressive loads. Thurston[1] has obtained, by the method of Lagrange's multiplier, the upper and lower bounds of critical loads of rectangular sandwich panels with clamped edges under uniaxial load and normal pressure. He has indicated that lower bound solutions are sufficiently close to experimental values. Ericksen and March[2] have obtained approximate buckling loads under uniaxial compression for rectangular sandwich panels by assuming a single sine wave for lateral deflection. Conway[3] has calculated deflection and moments of solid plates with clamped edges by using the Marcus method. Guest and Solvey[4] have extended March's method to obtain the critical loads of honeycomb sandwich panels with clamped edges under biaxial compressions. There seems to be no work published on the calculation of moments of rectangular sandwich panels with clamped edges. It is the purpose of the present study to develop a method of calculating these moments.

Method of Analysis

Hoff[5] has obtained the differential equations of equilibrium by minimizing the potential energies of the panel, including the shear energy of the core:

$$D_f\nabla^6 W_b - (D_f + D_0)\frac{G_c'C}{D_0}\nabla^4 W_b = \left(\nabla^2 - \frac{G_c'C}{D_0}\right)(p - N_x W_{b_{xx}} - N_y W_{b_{yy}}) \quad (1)$$

Considering the case where D_f is negligible, Eq. (1) reduces to

$$\nabla^4 W_b + \left(\frac{G_c'C}{D_0} - \nabla^2\right)\left(\frac{N_x}{G_c'C}W_{b_{xx}} + \frac{N_y}{G_c'C}W_{b_{yy}}\right) = \left(\frac{G_c'C}{D_0} - \nabla^2\right)\frac{p}{G_c'C} \quad (2)$$

This differential equation is applied to a rectangular panel with simply supported edges for the following three loadings: 1) under biaxial compressions (N_x, N_y) and uniform lateral load, 2) under biaxial compressions with no lateral load but with edge moment M_x applied at $x = \pm a/2$, and 3) under biaxial compressions with no lateral load but with edge moment M_y applied at $y = \pm b/2$. The solutions for these three loadings are first obtained. Then the three solutions are superposed to obtain the solution for clamped edges by equating the slope of W_b at the edges to zero. This determines the edge-moment distributions. Since the differential equation is linear, the solutions may be superposed.

Under loading (1), let

$$W_{b_{ss}} = \frac{p}{24D_0}(x^4 - 2ax^3 + a^3x) + \sum_{n=1}^{\infty} Y_m \sin\phi_m x \quad (3)$$

where $\phi_m = m\pi/a$ and Y_m is a function of y only.

From Ref. 6,

$$W_{s_{ss}} = -(D_0/G_c'C)\nabla^2 W_{b_{ss}} \quad (4)$$

Substituting Eq. (3) into Eq. (2) and simplifying

$$\frac{d^4Y_m}{dy^4}\left(1 - \frac{N_y}{G_c'C}\right) - \frac{d^2Y_m}{dy^2}\left(2\phi_m^2 - \frac{N_x}{G_c'C}\phi_m^2 - \frac{N_y}{G_c'C}\phi_m^2 - \frac{N_y}{D_0}\right) + Y_m\left(\phi_m^4 - \frac{N_x}{D_0}\phi_m^2 - \frac{N_x}{G_c'C}\phi_m^4\right) - \frac{4}{m\pi}\frac{p}{D_0}\frac{N_x}{G_c'C}\left(1 + \frac{a^2}{m^2\pi^2}\frac{G_c'C}{D_0}\right) = 0 \quad (5)$$

This is a linear differential equation of the fourth order. Its solution is

$$Y_m = Ae^{\theta_{1m}y} + Be^{-\theta_{1m}y} + Ce^{\theta_{2m}y} + De^{-\theta_{2m}y} + K_3/\theta_{1m}^2\theta_{2m}^2 \quad (6)$$

where

$$\theta_{1m} = \left[\frac{K_1 + (K_1^2 - 4K_2)^{1/2}}{2}\right]^{1/2}$$

$$\theta_{2m} = \left[\frac{K_1 - (K_1^2 - 4K_2)^{1/2}}{2}\right]^{1/2}$$

$$\left.\begin{aligned} K_1 &= \frac{[2\phi_m^2 - N_x/G_c'C\phi_m^2 - N_y/G_c'C\phi_m^2 - N_y/D_0]}{(1 - N_y/G_c'C)} \\ K_2 &= \frac{[\phi_m^4(1 - N_x/G_c'C) - N_x/D_0\phi_m^2]}{(1 - N_y/G_c'C)} \\ K_3 &= \frac{4/m\pi\ P/D_0\ N_x/G_c'C(1 + a^2G_c'C/m^2\pi^2D_0)}{(1 - N_y/G_c'C)} \end{aligned}\right\} \quad (7)$$

The boundary conditions are $W_T = 0$, and the sectional bending moment is zero at the four edges. The constants are solved from these boundary conditions, and the resulting deflection equation becomes

$$W_{b_{ss}} = \sum_{m=1,3,}^{\infty}\frac{4p}{m\pi D_0\phi_m^4}\frac{(-1)^{(m-1)/2}}{(1 - N_x/G_c'C - N_x/D_0\phi_m^2)} \times \left[1 - \frac{\theta_{2m}^2}{\theta_{2m}^2 - \theta_{1m}^2}\frac{\cosh\theta_{1m}y}{\cosh\theta_{1m}b/2} + \frac{\theta_{1m}^2}{\theta_{2m}^2 - \theta_{1m}^2}\frac{\cosh\theta_{2m}y}{\cosh\theta_{2m}b/2}\right]\cos\phi_m x \quad (8)$$

$$W_{T_{ss}} = \sum_{m=1,3,}^{\infty}\frac{4p}{m\pi D_0\phi_m^4}\frac{(-1)^{(m-1)/2}}{(1 - N_x/G_c'C - N_x/D_0\phi_m^2)} \times \left[\left(1 + \frac{D_0\phi_m^2}{G_c'C}\right) - \frac{\theta_{2m}^2\{1 + D_0/G_c'C(\phi_m^2 - \theta_{1m}^2)\}}{\theta_{2m}^2 - \theta_{1m}^2} \times \frac{\cosh\theta_{1m}y}{\cosh\theta_{1m}b/2} + \frac{\theta_{1m}^2\{1 + D_0/G_c'C(\phi_m^2 - \theta_m^2)\}}{\theta_{2m}^2 - \theta_{1m}^2} \times \frac{\cosh\theta_{2m}y}{\cosh\theta_{2m}b/2}\right]\cos\phi_m x \quad (9)$$

Under loading (2) with the edge moment expressed as

$$M_{x_{x=\pm(a/2)}} = \sum_{n=1,3,}^{\infty} F_n \sin\beta_n\left(y - \frac{b}{2}\right)$$

the deflection has been found by similar procedure to be

$$W_{b1} = \sum_{n=1,3,}^{\infty}\frac{F_n}{D_0}(-1)^{(n-1)/2} \times \frac{1 + D_0/G_c'C(\beta_n^2 - \theta_{a2n}^2)}{(\theta_{a2n}^2 - \theta_{a1n}^2)\{1 + D_0/G_c'C\beta_n^2(1 - \mu)\}\cosh\theta_{a1n}a/2} \times \left[\cosh\theta_{a1n}x - \frac{\{1 + D_0/G_c'C(\beta_n^2 - \theta_{a1n}^2)\}\cosh\theta_{a1n}a/2}{\{1 + D_0/G_c'C(\beta_n^2 - \theta_{a2n}^2)\}\cosh\theta_{a2n}a/z} \times \cosh\theta_{a2n}\right] x\cos\beta_n y \quad (10)$$

where

$$\theta_{a1n} = \left(\frac{K_{a1n} + (K_{a1n}^2 - 4K_{a2n})^{1/2}}{2}\right)^{1/2}$$

$$\theta_{a2n} = \left(\frac{K_{a1n} - (K_{a1n}^2 - 4K_{a2n})^{1/2}}{2}\right)^{1/2}$$

and

$$K_{a1n} = \frac{2\beta_n^2 - \beta_n^2(N_x/G_c'C + N_y/G_c'C) - N_x/D_0}{1 - N_x/G_c'C}$$

$$K_{a2n} = \frac{\beta_n^4 - \beta_n^2 N_y/D_0 - \beta_n^4 N_y/G_c'C}{1 - N_x/G_c'C}$$

Similarly, under loading (3), with the edge moment expressed as

$$M_{y_{y=\pm(b/2)}} = \Sigma E_m \sin\phi_m(x - a/2)$$

the deflection has been obtained as

$$W_{b2} = \sum_{m=1,3\ldots}^{\infty} \frac{E_m}{D_0}(-1)^{(m-1)/2} \times \frac{1 + D_0/G_c'C(\phi_m^2 - \theta_{2m}^2)}{(\theta_{2m}^2 - \theta_{1m}^2)\{1 + D_0/G_c'C\phi_m^2(1-\mu)\}\cosh\theta_{1m}b/2} \times \left[\cosh\theta_{1m}y - \frac{\{1 + D_0/G_c'C(\phi_m^2 - \theta_{1m}^2)\}\cosh\theta_{1m}b/2}{\{1 + D_0/G_c'C(\phi_m^2 - \theta_{2m}^2)\}\cosh\theta_{2m}b/2} \times \cosh\theta_{2m}y\right]\cos\phi_m x \quad (11)$$

In order to satisfy the condition of clamped edges,

$$\left.\frac{\partial W_{b_{ss}}}{\partial x}\right|_{x=a/2} + \left(\frac{\partial W_{b1}}{\partial x} + \frac{\partial W_{b2}}{\partial x}\right)_{x=a/2} = 0 \quad (12)$$

$$\left.\frac{\partial W_{b_{ss}}}{\partial y}\right|_{y=b/2} + \left(\frac{\partial W_{b1}}{\partial y} + \frac{\partial W_{b2}}{\partial y}\right)_{y=b/2} = 0 \quad (13)$$

From these two equations, E_m's and F_m's are determined.

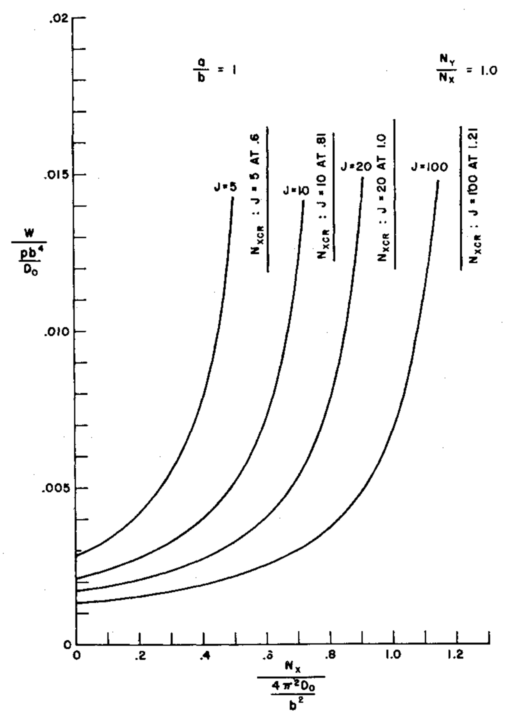

Fig. 1 Deflection at center of panel vs edge compression.

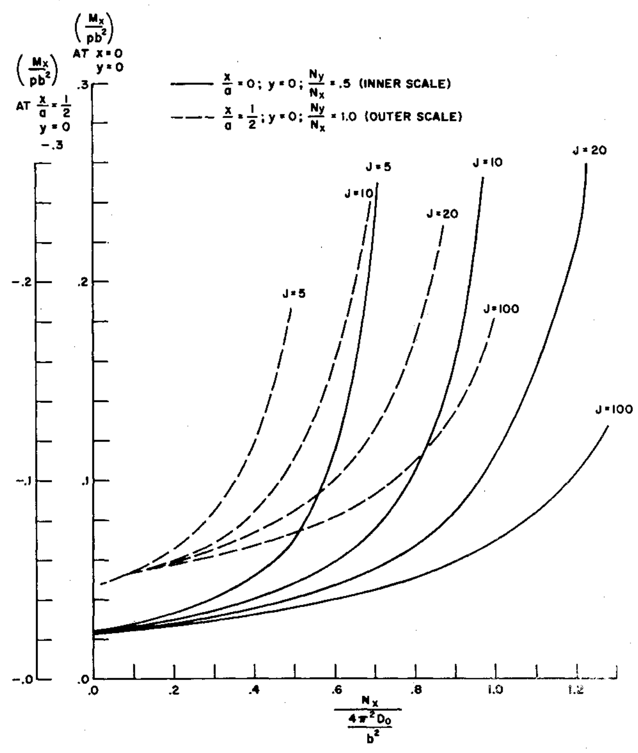

Fig. 2 Bending moment vs edge compression of a square panel.

Then solutions of W, M_x, M_y are obtained in terms of infinite series. It can be shown that these series are convergent.[7] However, the rapidity of convergency varies with core rigidity. It has been found that in the present calculation of deflections and moments, the rate of convergency decreases as the parameter $J(= b^2G_c'C/\pi^2D_0)$ decreases and/or N_y/N_x decreases. The values of moments converge much more slowly near the edge than at the center of the panel. In the present calculations, 30 to 40 terms have been used.

Conclusions

The lateral deflection calculated is plotted against axial compression in Fig. 1. The critical axial load occurs where the slope of this deflection curve approaches vertical. The critical loads given by the present method agree well with those by Thurston, Ericksen and March, and Guest and Solvey. The calculated moments at certain points of a square panel with various core rigidities are shown in Fig. 2. These calculations may be similarly made for other points on the panel.

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